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ON THE BINARY QUADRATIC DIOPHANTINE EQUATION $x^2 - 5xy + y^2 + 18x = 0$

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ABSTRACT

The binary quadratic equation $x^2 - 5xy + y^2 + 18x = 0$ representing hyperbola is considered and analysed for its integer points. A few interesting relations satisfied by x and y are exhibited.

Keywords: Binary quadratic, Hyperbola, Integer solutions.

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INTRODUCTION:

The binary quadratic equation offers an unlimited field for research because of their variety [1-5]. In this context, one may also refer [6-20]. This communication concerns with yet another interesting binary quadratic equation $x^2 - 5xy + y^2 + 18x = 0$ for determining its infinitely many non-zero integral solutions. Also a few interesting relations among the solutions are presented.

2. METHOD OF ANALYSIS: The hyperbola under consideration is

$$x^2 - 5xy + y^2 + 18x = 0 \qquad (1)$$

To start with, it is seen that (1) is satisfied by the following pairs of integers (18,0),(6,6),(6,24)(-18,-90),(96,24),(450,-90). However, we have other choices of solutions satisfying (1) and they are illustrated below:

Treating (1) as a quadratic in x and solving for x, we get

$$x = \frac{1}{2}\left[(5y - 18) \pm \sqrt{21y^2 - 180y + 324}\right] \qquad (2)$$

Let $\alpha^2 = 21y^2 - 180y + 324$ (3)

Substituting $y = \frac{Y+90}{21}$ (4)

in (3), we have

$Y^2 = 21\alpha^2 + 1$

whose general solution is given by

$$Y_n = \frac{1}{2}\left[\left(55+12\sqrt{21}\right)^{n+1} + \left(55-12\sqrt{21}\right)^{n+1}\right] \tag{5}$$

$$\alpha_n = \frac{1}{2\sqrt{21}}\left[\left(55+12\sqrt{21}\right)^{n+1} - \left(55-12\sqrt{21}\right)^{n+1}\right] \tag{6}$$

From (4) and (5), we have

$$y_n = \frac{6}{7}\left[\left(55+12\sqrt{21}\right)^{n+1} + \left(55-12\sqrt{21}\right)^{n+1}\right] + \frac{30}{7} \tag{7}$$

Substituting (6) and (7) in (2) and taking the positive sign, the corresponding integer solutions to (1) are given by

$$x_n = \frac{15}{7}\left[\left(55+12\sqrt{21}\right)^{n+1} + \left(55-12\sqrt{21}\right)^{n+1}\right] + \frac{12}{7} + \frac{9}{\sqrt{21}}\left[(55+12\sqrt{21})^{n+1} - (55-12\sqrt{21})^{n+1}\right],$$

$n = 1,3,5,\ldots\ldots$

$$y_n = \frac{6}{7}\left[\left(55+12\sqrt{21}\right)^{n+1} + \left(55-12\sqrt{21}\right)^{n+1}\right] + \frac{30}{7}, \quad n = 1,3,5,\ldots\ldots$$

PROPERTIES:

- $7y_{2n+1} - 18$ is a Nasty Number
- $\frac{1}{6}\left[7y_{3n+2} + 21y_n - 120\right]$ is a Cubical integer
- $\frac{1}{18}\left[21y_{4n+3} - 98y_n^2 - 840y_n + 1746\right]$ is a Bi-quadratic integer
- $126x_{2n+1} - 1339716\ x_{2n-1} - 302505\ y_{2n-1} = 35932890$
- $7x_{2n+3} - 1071193207\ x_{2n-1} + 223571040\ y_{2n-1} = 878169595$
- $7y_{2n+1} - 18480\ x_{2n-1} - 3857\ y_{2n-1} = -15150$
- $y_{2n+3} - 31938720\ x_{2n-1} + 6665999\ y_{2n-1} = -26183520$
- $x_{2n-1} - 12098\ x_{2n+1} + x_{2n+3} = -20736$
- $y_{2n-1} - 12098\ y_{2n+1} + y_{2n+3} = -51840$

Some numerical examples are presented below:

n	x_n	y_n
1	49686	10374
3	601080486	125452806

Also, taking the negative sign in (2), the other set of solutions to (1) is given by

$$x_n = \frac{15}{7}\left[\left(55+12\sqrt{21}\right)^{n+1} + \left(55-12\sqrt{21}\right)^{n+1}\right] + \frac{12}{7} - \frac{9}{\sqrt{21}}\left[(55+12\sqrt{21})^{n+1} - (55-12\sqrt{21})^{n+1}\right],$$

$n = 1,3,5,\ldots\ldots$

$$y_n = \frac{6}{7}\left[\left(55+12\sqrt{21}\right)^{n+1} + \left(55-12\sqrt{21}\right)^{n+1}\right] + \frac{30}{7} \quad , \qquad n = 1,3,5,\ldots\ldots$$

PROPERTIES:

- $126\, x_{2n+1} - 332640\; y_{2n-1} - 3857\; x_{2n-1} = -716247$
- $126\, x_{2n+3} - 574896960\; y_{2n-1} - 6665999\; x_{2n-1} = -12545280$

Alternatively, treating (1) as a quadratic in y and solving for y, we get

$$y = \frac{1}{2}\left[5x \pm \sqrt{21x^2 - 72x}\right] \tag{8}$$

Let $\alpha^2 = 21x^2 - 72x$ (9)

Substituting $x = \dfrac{X + 36}{21}$ (10)

in (9), we have

$$X^2 = 21\alpha^2 + 1$$

whose general solution is given by

$$X_n = \frac{1}{2}\left[\left(55+12\sqrt{21}\right)^{n+1} + \left(55-12\sqrt{21}\right)^{n+1}\right] \tag{11}$$

$$\alpha_n = \frac{1}{2\sqrt{21}}\left[\left(55+12\sqrt{21}\right)^{n+1} - \left(55-12\sqrt{21}\right)^{n+1}\right] \tag{12}$$

From (10) and (11), we have

$$x_n = \frac{6}{7}\left[\left(55+12\sqrt{21}\right)^{n+1} + \left(55-12\sqrt{21}\right)^{n+1}\right] + \frac{12}{7} \tag{13}$$

Substituting (12) and (13) in (8) and taking the positive sign, the corresponding integer solutions to (1) are given by

$$x_n = \frac{6}{7}\left[\left(55+12\sqrt{21}\right)^{n+1} + \left(55-12\sqrt{21}\right)^{n+1}\right] + \frac{12}{7} \quad , \quad n = 0,2,4,\ldots\ldots$$

$$y_n = \frac{15}{7}\left[\left(55+12\sqrt{21}\right)^{n+1} + \left(55-12\sqrt{21}\right)^{n+1}\right] + \frac{30}{7} + \frac{9}{\sqrt{21}}\left[(55+12\sqrt{21})^{n+1} - (55-12\sqrt{21})^{n+1}\right],$$

$n = 0,2,4,\ldots\ldots$

PROPERTIES:

- $x_{2n+2} - 2620\; y_{2n} - 551\, x_{2n} = -10368$
- $x_{2n+4} - 31938720\; y_{2n} - 6665999\; x_{2n} = -125452800$
- $y_{2n+2} - 12649\; y_{2n} + 47520\; x_{2n} = -49680$
- $y_{2n+4} - 15302761\; y_{2n} + 31938720\; x_{2n} = -601080480$
- $x_{2n+4} - 12098\; x_{2n+2} + x_{2n} = -20736$
- $y_{2n+4} - 12098\; y_{2n+2} + y_{2n} = -51840$

Also, taking the negative sign in (8), the other set of solutions to (1) is given by

$$x_n = \frac{6}{7}\left[\left(55+12\sqrt{21}\right)^{n+1} + \left(55-12\sqrt{21}\right)^{n+1}\right] + \frac{12}{7} \quad , \quad n = 0,2,4,\ldots\ldots$$

$$y_n = \frac{15}{7}\left[\left(55+12\sqrt{21}\right)^{n+1}+\left(55-12\sqrt{21}\right)^{n+1}\right]+\frac{30}{7}-\frac{9}{\sqrt{21}}\left[(55+12\sqrt{21})^{n+1}-(55-12\sqrt{21})^{n+1}\right],$$

$n = 0,2,4,\ldots\ldots$

PROPERTIES:

- $y_{2n+2} - 2640\, x_{2n} - 551\, y_{2n} = -13775$
- $y_{2n+4} - 31938720\; x_{2n} + 6665999\; y_{2n} = -26183520$

In addition to the above two choices of solutions, we have an another pattern as shown below:

Introducing the linear transformations

$$x = u + v \quad , \quad y = u - v \tag{14}$$

in (1), it is written as

$$Y^2 = 21X^2 - 108 \tag{15}$$

where

$$Y = 7v - 9 \quad , \quad X = u - 3 \tag{16}$$

The smallest positive integer solution of (15) is $(X_0,Y_0)=(3,9)$

Let $\left(\tilde{X}_n, \tilde{Y}_n\right)$ be the general solution of the Pellian equation $Y^2=21X^2+1$

where $\tilde{X}_n = \frac{1}{2\sqrt{21}}\left[\left(55+12\sqrt{21}\right)^{n+1}-\left(55-12\sqrt{21}\right)^{n+1}\right]$

$$\tilde{Y}_n = \frac{1}{2}\left[\left(55+12\sqrt{21}\right)^{n+1}+\left(55-12\sqrt{21}\right)^{n+1}\right]$$

Applying the lemma of Bramhagupta between the solutions $\left(X_0, Y_0\right)$ and $\left(\tilde{X}_n, \tilde{Y}_n\right)$, the values of X and Y satisfying (15) are given by

$$X_{n+1} = X_0\tilde{Y}_n + Y_0\tilde{X}_n$$

$$Y_{n+1} = Y_0\tilde{Y}_n + 21X_0\tilde{X}_n$$

In view of (16) and (14), the values of x and y are given by

$$x_{n+1} = \frac{1}{7}\left[30\tilde{Y}_n + 126\tilde{X}_n + 30\right], n = 0,2,4,\ldots$$

$$y_{n+1} = \frac{1}{7}\left[12\tilde{Y}_n + 30\right], n = 0,2,4,\ldots$$

Note that the value of x and y are integers when n is even. Thus, the integer values of x and y satisfying (1) are represented by,

$$x_{2n+1} = \frac{15}{7} f_n + \frac{9}{\sqrt{21}} g_n + \frac{30}{7}$$

$$y_{2n+1} = \frac{6}{7} f_n + \frac{12}{7}$$

where $f_n = \left[\left(55+12\sqrt{21}\right)^{2n+1}+\left(55-12\sqrt{21}\right)^{2n+1}\right]$

$$g_n = \left[\left(55+12\sqrt{21}\right)^{2n+1}-\left(55-12\sqrt{21}\right)^{2n+1}\right]$$

CONCLUSION

As the binary quadratic equations representing hyperbolas are rich in variety, one may consider other forms of hyperbolas and search for their non-trivial distinct integral solutions along with the corresponding properties.

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