International journal of Engineering Research-Online
A Peer Reviewed International Journal
Articles available online http://www.ijoer.in

Vol.1., Issue.4., 2013

RESEARCH ARTICLE

ISSN:2321-7758

ON TERNARY QUADRATIC EQUATION $x^2 + y^2 - xy = 7z^2$

M.A.GOPALAN, S.VIDHYALAKSHMI , S.NIVETHITHA
Department of Mathematics, Shrimati Indira Gandhi College, Tiruchirappalli, India

Article Received: 26/10/2013 | Article Revised on: 26/12/2013 | Article Accepted on:27/12/2013

S.NIVETHITHA

SCAN QR CODE FOR MORE DETAILS

ABSTRACT

The ternary quardratic diophantine equation given by $x^2 + y^2 - xy = 7z^2$ is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

KEY WORDS: Ternary quadratic, Integral solutions.

M.SC 2000 mathematics subject classification:

INTRODUCTION

The ternary quadratic diophantine equation offers an unlimited field for research because of their variety [1-3]. For an extensive review of various problems, one may refer [4-13]. This communication concerns with yet another interesting ternary quadratic equation $x^2 + y^2 - xy = 7z^2$ for determining its infinitely many non-zero integral solutions. Also a few interesting relations among the solutions have been presented.

NOTATIONS USED

- $t_{m,n}$ = Polygonal number of rank n with sides m.,
- p_n^5 = Pentagonal pyramidal number of rank n.

- p_n^3 = Tetrahedral number of rank n .

2. METHOD OF ANALYSIS

The ternary Quadratic Diophantine Equation to be solved for its non-zero distinct integral solutions is

$$x^2 + y^2 - xy = 7z^2 \quad (1)$$

The substitution of linear transformations

$$x = u + v, y = u - v (u \neq v \neq 0) \quad (2)$$

in (1) leads to $u^2 + 3v^2 = 7z^2$ (3)

(3) is solved through different approaches and thus different patterns of solutions of (1) are presented below.

2.1 PATTERN-I

Write 7 as $7 = (2 + i\sqrt{3})(2 - i\sqrt{3})$ (4)

Assume $z = a^2 + 3b^2$ (5)

where a,b are non-zero distinct integers.

Using (4) and (5) in (3) and employing the method of factorization, define

$$(u + i\sqrt{3}v) = (2 + i\sqrt{3})(a + i\sqrt{3}b)^2$$

Equating real and imaginary parts, we get

$$u = u(a,b) = 2a^2 - 6b^2 - 6ab$$

$$v = v(a,b) = a^2 - 3b^2 + 4ab$$

Substituting u and v values in (2), we get

$$x = x(a,b) = 3a^2 - 9b^2 - 2ab \quad (6)$$

$$y = y(a,b) = a^2 - 3b^2 - 10ab \quad (7)$$

Thus (5), (6) and (7) represent non-zero distinct integral solutions of (1) in two parameters.

A few interesting properties observed are as follows:

- $x(2, a+1) + 18t_{3,a} + 1 \equiv 0 (\text{mod } 13)$
- $y(1, 3b) - 54t_{3,b} - 1 \equiv 0 (\text{mod } 3)$
- $x(2a, 1) + y(1, a) - 18t_{3,a} + 8 \equiv 0 (\text{mod } 23)$

2.2 PATTERN II

Introducing the linear transformations

$$z = X + 3T, v = X + 7T \quad (8)$$

in (3) leads to $X^2 = W^2 + 21T^2$ (9)

which is satisfied by,

$$T = 2rs, X = 21r^2 + s^2, W = 21r^2 - s^2$$

Substituting the above values of T,x,w in (8) & (2), the corresponding non-zero distinct integral solutions are,

$$x = x(r,s) = 63r^2 - s^2 + 14rs$$

$$y = y(r,s) = 21r^2 - 3s^2 - 14rs$$

$$z = z(r,s) = 21r^2 + s^2 + 6rs$$

PROPERTIES

- $x(r,1)-28t_{3,r}+1-t_{106,r}$ is a perfect square
- $y(r+1,1)-42t_{3,r}+3\equiv 0(\bmod 35)$
- $y(r,r+2)-44t_{3,r}+2\equiv 0(\bmod 10)$
- $y(1,s+2)-4t_{3,s}-2\equiv 0(\bmod 5)$

Note:

suppose instead of (8), we have

$$z=X-3T, v=X-7T$$

Thus ,the corresponding non-zero distinct integral solutions to (1) are given by

$$x=x(r,s)=63r^2-s^2-14rs$$

$$y=y(r,s)=21r^2-3s^2+14rs$$

$$z=z(r,s)=21r^2+s^2-6rs$$

PROPERTIES

- $x(s+1,8)-126t_{3,s}+15\equiv 0(\bmod 49)$
- $x(r,6)+y(3,r+1)-120t_{3,r}-84\equiv 0(\bmod 108)$
- $2x(s,4)+z(2,s)-246t_{3,s}-52\equiv 0(\bmod 207)$

2.3 PATTERN III

Write (3) in the form of ratio as,

$$\frac{u+2z}{z+v}=\frac{3(z-v)}{u-2z}=\frac{A}{B}, B\neq 0 \quad (11)$$

which is equivalent to the system of equations

$$Bu-Av+(2B-A)z=0$$

$$Au+3Bv-(2A+3B)z=0$$

Applying the method of cross multiplication between the above system of equations, we have

$$u=2A^2+6AB-6B^2, v=-A^2+4AB+3B^2 \quad (12)$$

$$z=3B^2+A^2 \quad (13)$$

Substitute (12) in (2) we have

$$x=x(A,B)=A^2+10AB-3B^2 \quad (14)$$

$$y=y(A,B)=3A^2+2AB-9B^2$$

Thus (13) and (14) represent non-zero distinct integral solutions of (1) in two parameters.

PROPERTIES

- $2x(5,a+2)+t_{178,a}-164t_{3,a}$ is a perfect square.
- $y(2,b)+t_{20,b}\equiv 0(\bmod 4)$
- $x(a,1)+y(2,a)+t_{18,a}-2\equiv 0(\bmod 7)$

2.4 PATTERN IV

(3) is also written in the form of ratio as

$$\frac{u+2z}{3(z+v)}=\frac{(z-v)}{u-2z}=\frac{A}{B}, B\neq 0$$

Following the procedure as in pattern III, the corresponding Integral solutions of (1) as given by

$$x=x(A,B)=-3A^2-10AB+B^2$$

$$y=y(A,B)=-9A^2+3B^2-2AB$$

$$z=z(A,B)=-(B^2+3A^2)$$

PROPERTIES

- $y(8,b+1)-2t_{5,b}+5\equiv 0(\bmod 8)$
- $x(a+2,3)+z(a,5)+12t_{3,a}-32\equiv 0(\bmod 34)$
- $2x(a,8)+y(2,a)+6t_{3,a}-92\equiv 0(\bmod 161)$

2.5 PATTERN V

Equation (3) is written as

$$7z^2-3v^2=u^2=u^2*1$$

Write 1 as

$$1=\frac{(\sqrt{7}+\sqrt{3})(\sqrt{7}-\sqrt{3})}{4} \quad (16)$$

Assume $u=7a^2-3b^2$ (17)

Substituting (16) & (17) in (15) and employing the method of factorization, define

$$\sqrt{7}z+\sqrt{3}v=\frac{1}{2}\left[7\sqrt{7}a^2+3\sqrt{7}b^2+14\sqrt{3}ab+7\sqrt{3}a^2+3\sqrt{3}b^2+6\sqrt{7}ab\right]$$

Which is satisfied by

$$z=\frac{1}{2}\left[7a^2+3b^2+6ab\right]$$

$$v=\frac{1}{2}\left[14ab+7a^2+3b^2\right]$$

using the values of u & v in (2) we have

$$x=\frac{1}{2}\left[14ab+21a^2-3b^2\right]$$

$$y=\frac{1}{2}\left[14ab-7a^2+9b^2\right]$$

As our aim is to find integral solutions.It is possible to choose a and b so that x and y are integers.

Case (i)

Let $a=2A, b=2B$

The corresponding integer solutions are

$$x = x(A,B) = 42A^2 - 6B^2 + 28AB$$
$$y = y(A,B) = 14A^2 - 18B^2 - 28AB$$
$$z = z(A,B) = 14A^2 + 6B^2 + 12AB$$

PROPERTIES

- $2x(a,5) - 168t_{3,a} + 104 \equiv 0 \pmod{196}$
- $3x(3,a) + 2y(1,a) + t_{110,a} - 49 \equiv 0 \pmod{53}$
- $y(a,8) + 2z(3,a) - t_{54,a} + 11 \equiv 0 \pmod{127}$

Case (ii):

Let $a = 3n+1, b = 3n-1$.

The corresponding integer solutions are

$$x = 144n^2 + 72n + 2$$
$$y = -72n^2 + 48n + 6$$
$$z = 72n^2 + 12n + 2$$

3. REMARKABLE OBSERVATIONS

1. If the non-zero integer triple (x_0, y_0, z_0) is any solutions of (1) then the triple (x_n, y_n, z_n) Where

$$x_n = \frac{2^{n+1}\sqrt{21}u_0 + \sqrt{21}A_n v_0 + 7B_n z_0}{2\sqrt{21}}$$
$$y_n = \frac{2^{n+1}\sqrt{21}u_0 - \sqrt{21}A_n v_0 - 7B_n z_0}{2\sqrt{21}}$$
$$z_n = \frac{1}{2\sqrt{21}}\left[3B_n v_0 + \sqrt{21}A_n z_0\right]$$

also satisfies (1).

Here $A_n = (5+\sqrt{21})^n + (5-\sqrt{21})^n$,

$B_n = (5+\sqrt{21})^n - (5-\sqrt{21})^n$

2. If the non-zero integer triple (x_0, y_0, z_0) is any solutions of (1) then the triple (x_n, y_n, z_n) Where also satisfies (1).

$$x_n = \frac{3\sqrt{7}A_n u_0 + 21B_n z_0 + 6\sqrt{7}v_0}{6\sqrt{7}}$$
$$y_n = \frac{3\sqrt{7}A_n u_0 + 21B_n z_0 - 6\sqrt{7}v_0}{6\sqrt{7}}$$
$$z_n = \frac{1}{6\sqrt{7}}\left[3B_n u_0 + 3\sqrt{7}A_n z_0\right]$$

Here

$$A_n = (8+3\sqrt{7})^n + (8-3\sqrt{7})^n,$$
$$B_n = (8+3\sqrt{7})^n - (8-3\sqrt{7})^n.$$

3. $7\left[\left(\frac{p_x^5}{t_{3,x}}\right)^2 + \left(\frac{p_y^5}{t_{3,y}}\right)^2 - \frac{p_x^5}{t_{3,x}}\cdot\frac{p_y^5}{t_{3,y}}\right].$ is a perfect square

4. $\left(\frac{p_x^5}{t_{3,x}} + \frac{p_y^5}{t_{3,y}}\right)^2 - 3\frac{p_x^5}{t_{3,x}}\frac{p_y^5}{t_{3,y}} \equiv 0 \pmod 7$

5. $\left(\frac{p_x^5}{t_{3,x}} + \frac{p_y^5}{t_{3,y}}\right)^2 - 3\frac{p_x^5}{t_{3,x}}\frac{p_y^5}{t_{3,y}} = 7\left(\frac{3p_{z-2}^3}{t_{3,z-2}}\right)^2$

6. If x ,y are taken as the generators of a Pythagorean triangle, then the Hypotenuse is congruent to the product of its generators under module 7.

7. Consider x and y to be the length and breath of a rectangle R, whose

Area = A

Perimeter = P and

Length of the diagonal= L

Then, it is noted that

1.$L^2 - A \equiv 0 \pmod 7$

2.$42(L^2 - A)$

3.$P^2 - 12A \equiv 0 \pmod{28}$

4. CONCLUSION

In this paper, we have obtained infinitely many non-zero distinct integral points on the homogeneous cone given by $x^2 + y^2 - xy = 7z^2$

To conclude, one many search for the integral points on other choices of curves, namely, hyperboloid, paraboloid, and hyperbolic paraboloid and so on.

REFERENCES

[1]. L.E.Dickson, History of Theory of Numbers, vol.2,Chelsea Publishing Company, New York, 1952.

[2]. L.J.Mordell, Diophantine Equations, Academic press, London, 1969.

[3]. Andre weil, Number theory: An Approach through history: from hammurapi to legendre / Andre weil: Boston (Birkahasuser boston),1983.

[4].M.A.Gopalan, S.Vidhyalakshmi and S.Devibala, Integral solutions of $ka(x^2 + y^2) + bxy = 4k\alpha^2 z^2$, Bulletin of pure and applied sciences, vol.25E, No:2, (2006), 401 – 406.

[5]. M.A.Gopalan, S.Vidhyalakshmi and S.Devibala, Integral solutions of $7x^2 + 8y^2 = 9z^2$, Pure and Applied Mathematika Sciences, Vol.LXVI, No:1-2, 2007, 83-86.

[6]. M.A.Gopalan and S.Vidhyalakshmi, An observation on $kax^2 + by^2 = cz^2$, Acta Cienica Indica Vol.XXXIIIM, No:1,2007,97-99.

[7]. M.A.Gopalan, Manju somanath and N.Vanitha, Integral solutions of $kxy + m(x+y) = z^2$, Acta Ciencia Indica, Vol.XXXIIIM, No:4, (2007) 1287 – 1290.

[8].M.A.Gopalan and V.Pondichelvi, On ternary Quadratic Equation $x^2 + y^2 = z^2 + 1$, Impact.J.Sci.Tech, Vol(2), No:2,2008, 55-58.

[9]. M.A.Gopalan and J.Kaligarani, On Ternary Quadratic Equation $x^2 + y^2 = z^2 + 8$, Impact J.Sci.Tech; Vol.(5), No:1,39-43. 2011

[10].M.A.Gopalan,S.Vidhyalakshmi, E.Premalatha, on ternary Quadratic Diophantine Equation $x^2 + 3y^2 = 7z^2$, Diophantus J.Math., 1(1) (2012), 51-57.

[11].M.A.Gopalan, S.Vidhyalakshmi, N.Thiruniraiselvi, observations on $x^2 + y^2 = 17z^2$, Diophantus J.Math, 1(2) (2012), 77-83.

[12].M.A.Gopalan,S.vidhyalakshmi, M.Manjula on ternary Quadratic Equation $x^2 - xy + y^2 = 19z^2$, Diophantus J.Math, 1(2) (2012), 85-91.

[13].S.Vidhyalakshmi,K.Lakshmi,M.A.Gopalan,Lattice points on the Elliptic paraboloid $16y^2 + 9z^2 = 4x$,Bessel J.Math.,3(2),2013,137-145.