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RESEARCH ARTICLE

ON CUBIC DIOPHANTINE EQUATION WITH THREE UNKNOWNS $x^2 - xy + y^2 = 19z^3$

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ABSTRACT
The non-homogeneous cubic equation with three unknowns represented by the diophantine equation $x^2 - xy + y^2 = 19z^3$ is analysed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special numbers are exbited.

KEYWORDS: Integral solutions, non-homogeneous cubic equation with three unknowns.
MSC mathematics Subject classification:11D25

Notations:

$t_{m,n}$: Polygonal number of rank n with size m

P_n^m : Pyramidal number of rank n with size m

J_n : Jacobsthal number of rank n

P_n : Pronic number of rank n

G_n : Gnomonic number of rank n

S_n : Star number of rank n

CP_n^m : Centered Pyramidal number of rank n with size m

$Ct_{m,n}$: Centered Polygonal number of rank n with size m

TT_n : Truncated tetrahedral number of rank n

TO_n : Truncated octahedral number of rank n

HO_n : Hauy octahedral number of rank n

H_n : Hex number of rank n

M_n : Mersenne number of rank n

HG_n : Hexagonal number of rank n

SO_n : Star number of rank n

INTRODUCTION

The Diophantine equations offer an unlimited field for research due to their variety[1-3]. In particular, one may refer [4-23] for cubic equations with three unknowns. This communication concerns with yet another interesting equation $x^2 - xy + y^2 = 19z^3$ representing non-homogeneous cubic with three unknowns for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

METHOD OF ANALYSIS:

The ternary non-homogeneous cubic equation to be solved for its distinct non-zero integral solution is

$$x^2 - xy + y^2 = 19z^3 \qquad (1)$$

Pattern:1

Introducing the linear transformations,

$$x = u + v, y = u - v \qquad (2)$$

in (1) leads to

$$u^2 + 3v^2 = 19z^3 \qquad (3)$$

Let $z = a^2 + 3b^2$ (4)

Write 19 as

$$19 = \frac{(7 + i3\sqrt{3})(7 - i3\sqrt{3})}{4} \qquad (5)$$

Using (4) and (5) in (3) and applying the method of factorization, define

$$u + i\sqrt{3}v = \frac{(7 + i3\sqrt{3})}{2}(a + i\sqrt{3}b)^3$$

Equating the real and imaginary parts, we get

$$u = \frac{1}{2}\left[7a^3 - 27a^2b - 63ab^2 + 27b^3\right] \qquad (6)$$

$$v = \frac{1}{2}\left[3a^3 + 21a^2b - 27ab^2 - 21b^3\right] \qquad (7)$$

Since our aim is to find an integer solutions, so substitute $a = 2A, b = 2B$ in (4), (6) and (7) the corresponding integral values of x, y, z satisfying (1) are obtained as,

$$x(A,B) = 40A^3 - 24A^2B - 360AB^2 + 24B^3$$

$$y(A,B) = 16A^3 - 192A^2B - 144AB^2 + 192B^3$$

$$z(A,B) = 4A^2 + 12B^2$$

Properties:

1 $x(2^n,1) - 40(-1)^{3n} = 12\left[10J_{3n} - 9J_{2n} - 90J_n - 2(-1)^{2n} - 30(-1)^n + 8\right]$

2. $y(\mathrm{A},1) - 16\left[P_A^3 - 15P_A + G_{2A}\right] \equiv 0 \pmod{13}$

3. $z(\mathrm{A},\mathrm{A}-1) - 4t_{10,A} \equiv 0 \pmod{3}$

4. $y(\mathrm{A},\mathrm{A}) + z(\mathrm{A},\mathrm{A}) - x(\mathrm{A},\mathrm{A}) + 32P_A^5 \equiv 0 \pmod{176}$

5. $6\ z(\mathrm{A},\mathrm{A}-1) - 3$ is a nasty number.

Pattern II

(3) can be written as,

$$u^2 + 3v^2 = 19z^3 \times 1 \tag{8}$$

Write 19and 1 as

$$19 = (4 + i\sqrt{3})(4 - i\sqrt{3}) \tag{9}$$

$$1 = \frac{(1 + i\sqrt{3})(1 - i\sqrt{3})}{4} \tag{10}$$

Substituting (4), (9) and (10) in (8) we get,

$$(u + i\sqrt{3}v)(u - i\sqrt{3}v) = (4 + i\sqrt{3})(4 - i\sqrt{3})(a + i\sqrt{3}b)^3 \frac{(1 + \mathrm{i}\sqrt{3})}{2}\frac{(1 - \mathrm{i}\sqrt{3})}{2}$$

Equating the real and imaginary parts, the values of u, v are given by

$$u(a,b) = \frac{1}{2}\left[a^3 - 45a^2b - 9ab^2 + 45b^3\right]$$

$$v(a,b) = \frac{1}{2}\left[5a^3 + 3a^2b - 45ab^2 - 3b^3\right]$$

Substituting these values in (2) the corresponding integral values of x, y, z satisfying (1) are obtained as,

$$x(a,b) = 3a^3 - 21a^2b - 27ab^2 + 21b^3$$

$$y(a,b) = -2a^3 - 24a^2b + 18ab^2 + 24b^3$$

$$z(a,b) = a^2 + 3b^2$$

Properties:

1. $x(1,b) - 42P_b^5 + 96t_{3,b} \equiv 3(\mathrm{mod}\,27)$
2. $y(1,b) - 48P_b^5 + 12t_{3,b} \equiv -2(\mathrm{mod}\,18)$
3. $x(1,2b) - 336P_b^5 + 552t_{3,b} \equiv 3(\mathrm{mod}\,234)$
4. $x(2b,1) - 126P_b^3 + 171t_{3,b} \equiv 24(\mathrm{mod}\,45)$
5. $y(a,1) + 2P_a^3 + 9t_{6,a} \equiv 11(\mathrm{mod}\,13)$

Pattern III:

Introducing the linear transformations

$$u = \alpha + 3T, v = \alpha - T \tag{11}$$

Substituting (11) in (3) we get,

$$4\alpha^2 + 12T^2 = 19z^3 \tag{12}$$

Let $z = a^2 + 12b^2$ (13)

Write 19 as,

$$19 = \frac{(8 + i\sqrt{12})(8 - i\sqrt{12})}{4} \tag{14}$$

Using (13) and (14) in (12), we get

$$2\alpha + i\sqrt{12}T = \frac{(8+i\sqrt{12})}{2}(a+i\sqrt{12}b)^3$$

Equating the real and imaginary parts, we obtained

$$2\alpha = 4(a^3 - 36ab^2) - 6(3a^2b - 12b^3)$$

$$2T = (a^3 - 36ab^2) + 8(3a^2b - 12b^3)$$

Hence the values of x, y, z satisfies (1) are given by

$$x(a,b) = 5(a^3 - 36ab^2) + 2(3a^2b - 12b^3)$$

$$y(a,b) = 2(a^3 - 36ab^2) + 16(3a^2b - 12b^3)$$

$$z(a,b) = a^2 + 12b^2$$

Properties:

1. $x(a,a) - y(a,a) + 69CP_a^6 = 0$
2. $x(a,1) + 15(CP_a^6 - CP_a^8) - Ct_{14,a} \equiv 176 (\bmod 181)$
3. $y(1,b) + 144CP_a^8 + 12S_b \equiv -60 (\bmod 72)$
4. $z(a^2, a(a-1)) = (t_{8,a})^2 - 3P_{a^2-1}$
5. $x(a,a+1) + 5TO_a + 12(TT_a - CP_a^{18}) + 30CP_a^{19} \equiv 6 (\bmod 8)$

Instead of (11), consider the linear transformation,

$$u = \alpha - 3T, v = \alpha + T$$

and write 19 as

$$19 = \frac{(14+i3\sqrt{12})(14-i3\sqrt{12})}{16}$$

Following the procedure as presented in the above Pattern the corresponding non-zero distinct integral solutions to (1) are obtained as

$$x(a,b) = 2(a^3 - 36ab^2) - 16(3a^2b - 12b^3)$$

$$y(a,b) = 14(36ab^2 - a^3) + 14(12b^3 - 3a^2b)$$

$$z(a,b) = a^2 + 12b^2$$

Pattern IV:

(1) can be written as

$$(2x - y)^2 + 3y^2 = 76z^3 \qquad (15)$$

One may write 76 as

$$76 = (8 + i2\sqrt{3})(8 - i2\sqrt{3}) \qquad (16)$$

substituting (16) and (4) in (15), employing the method of factorization, we have

$$(2x - y) + i\sqrt{3}y = (8 + i2\sqrt{3})(a + i\sqrt{3}b)^3$$

Equating the real and imaginary parts, we have

$$x(a,b) = 5(a^3 - 9ab^2) + 3(a^2b - b^3)$$

$$y(a,b) = 2(a^3 - 9ab^2) + 24(a^2b - b^3)$$

$$z(a,b) = a^2 + 3b^2$$

Properties:

1. $x(a,a)+3HO_a+6P_a^6+H_a-y(a,a)-z(a,a)\equiv 0(\bmod 2)$

2. $t_{10,a}-z(a,a-1)\equiv 0(\bmod 3)$

3. $4P_a^5+6H_a-y(a,1)\equiv 2(\bmod 4)$

4. $x(1,b)+46P_b-2P_b^{11}\equiv 2(\bmod 5)$

5. $6\left[M_{2a}-z(2^a,1)\right]$ is a nasty number.

REMARKABLE OBSERVATIONS:

1.Let (x_0,y_0,z_0) be the initial solution of (1)

Let
$$\left.\begin{aligned} x_1&=19^3x_0+h\\ y_1&=19^3y_0+h\\ z_1&=19^2z_0\end{aligned}\right\} \quad (17)$$

be the first solution of (1).

Substituting (17) in (1), we get

$$h=-19^3(x_0+y_0) \quad (18)$$

Using (18) in (17) we obtain the general solution as follows,

EVENORDERED SOLUTION:

$$x_{2n}=19^{6n}x_0,$$
$$y_{2n}=19^{6n}y_0,$$
$$z_{2n}=19^{4n}z_0, \quad \text{where } n=1,2,3...$$

ODDORDERED SOLUTION:

$$x_{2n-1}=-19^{3(2n-1)}y_0,$$
$$y_{2n-1}=-19^{3(2n-1)}x_0,$$
$$z_{2n-1}=-19^{2(2n-1)}z_0, \quad \text{where } n=1,2,3...$$

2.Let R be a rectangle with sides x,y such that

L =length of the rectangle,

A =Area of the rectangle,

P =Perimeter of the rectangle,

Then we have the following relations,

$$L^2\equiv A(\bmod 19)$$
$$P^2\equiv 12A(\bmod 19)$$

3.Employing the solutions (x,y,z) of(1), the following relations among the special polygonal ,pyramidal and some special numbers are obtained.

1. $\left(\dfrac{2P_{x-1}^8}{t_{3,2x-3}}\right)^2-\left(\dfrac{2P_{x-1}^8}{t_{3,2x-3}}\right)\left(\dfrac{P_y^3}{P_{y+1}}\right)+\left(\dfrac{P_y^3}{P_{y+1}}\right)^2=19\left(\dfrac{HG_{z^2}}{SO_z}\right)^3$

2. $\left(\frac{P_x^5}{t_{3,x}}\right)^2-\left(\frac{P_x^5}{t_{3,x}}\right)\left(\frac{HG_{y^2}}{SO_y}\right)+\left(\frac{HG_{y^2}}{SO_y}\right)^2=19\left(\frac{3P_z^3}{t_{3,y+1}}\right)^3$

3. $\left(\frac{P_x^3}{t_{3,x+1}}\right)^2-\left(\frac{P_x^3}{t_{3,x+1}}\right)\left(\frac{P_y^5}{t_{3,y}}\right)+\left(\frac{P_y^5}{t_{3,y}}\right)^2=19\left(\frac{P_z^5}{t_{3,z}}\right)^3$

4. $\left(\frac{6P_x^4}{HG_{x+1}}\right)^2-\left(\frac{6P_x^4}{HG_{x+1}}\right)\left(\frac{HG_y}{G_y}\right)+\left(\frac{HG_y}{G_y}\right)^2=19\left(\frac{6P_z^4}{t_{3,2z+1}}\right)^3$

CONCLUSION

It is worth to mention that instead of (5) and (10) one may write 19 and 1 as

$$19=\frac{(1+i5\sqrt{3})(1-i5\sqrt{3})}{4}$$

$$1=\begin{cases}\frac{(1+i4\sqrt{3})(1-i4\sqrt{3})}{49}\\ \frac{(11+i4\sqrt{3})(11-i4\sqrt{3})}{169}\\ \frac{(11+i5\sqrt{3})(11-i5\sqrt{3})}{196}\\ \frac{(13+i3\sqrt{3})(13-i3\sqrt{3})}{196}\\ \frac{(13+i8\sqrt{3})(13-i8\sqrt{3})}{169}\\ \frac{(23+i7\sqrt{3})(23-i7\sqrt{3})}{676}\end{cases}$$

respectively. Following the procedure as presented in PatternI and Pattern II , the other patterns of solutions to (1) are determined. To conclude one may search for other patterns of solutions and their related properties.

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